Model Reduction for Flexible Space Structures

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This paper presents the conditions under which modal truncation yields a near-optimal reduced-order model for a flexible structure. Next, a robust model reduction technique is developed to cope with the damping uncertainties typical of a flexible space structure. Finally, a flexible truss and the Control of Flexible Structures-1 mast are used to give realistic applications for the model reduction techniques studied in this paper.

Introduction

MODEL reduction is one of the important issues in the dynamic analysis of flexible structures. A model with a large number of degrees of freedom, although useful in a static analysis, can cause numerical difficulties, uncertainties, and high computational costs if used to study the dynamics of a system. The problem of model reduction has a long history, and among many techniques we mention balanced and modal truncation.[1–8] A related method is described in Ref. 12 that allows reduced-order model matching over a limited time or frequency interval; this can be useful for modeling unstable systems, or reduction in a specified frequency band of interest. Although these techniques are all comparatively simple, they do not necessarily give optimal results; Gawronski and Juang[11] present necessary and sufficient conditions for near-optimal model reduction in balanced or modal coordinates. The optimal reduction methods of Wilson[9] and Hyland and Bernstein[10] are, on the other hand, computationally very expensive. Furthermore, the references listed thus far were mainly concerned with reduction of general linear systems. Model reduction for the specific case of flexible structures is considered by Jonckheere,[6] Gregory,[5] and Skelton et al.[1,2,13] These authors have shown that, in the case of small damping and widely separated natural frequencies, the balanced and modal representations of a flexible structure are almost identical.

This paper presents the conditions under which modal truncation of a flexible structure yields a near-optimal reduced-order model based on a new generalization of the simple closed-form grammians obtained in Refs. 2 and 14 for flexible structures with rate measurements only. In particular, it is shown that modal and balanced reduction give very different results for the typical flexible space structure (FSS) case of closely spaced natural frequencies, with balanced reduction generally giving far better results. Another question addressed in the paper is the sensitivity of modeling error to variations in the damping of a structure and the development of a robust model reduction technique to cope with this. The great practical importance of this problem is a consequence of the fact that the damping level present in an FSS is usually poorly defined and difficult to identify in ground tests. Finally, a flexible truss and the Control of Flexible Structures-1 (COFS-1) mast are used to give realistic applications for the model reduction techniques studied in the paper.

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Problem Formulation

Consider an n-mode model for the structural dynamics of a modally damped, nongyroscopic, noncirculatory FSS with m actuators and p sensors, not necessarily collocated. This model can be written in modal form[15] as

$$\ddot{\boldsymbol{\eta}} + \mathrm{diag}\,(2\zeta_i\omega_i)\dot{\boldsymbol{\eta}} + \mathrm{diag}\,(\omega_i^2)\boldsymbol{\eta} = \hat{B}u \tag{1a}$$

$$y = \hat{C}_r\dot{\boldsymbol{\eta}} + \hat{C}_d\boldsymbol{\eta} \tag{1b}$$

where $\boldsymbol{\eta}$ is the vector of modal coordinates, and ω_i and ζ_i are the natural frequency and damping ratio of the ith mode, respectively. For the typical FSS (Ref. 16), the $\{\zeta_i\}$ are quite low (e.g., 0.005), and the $\{\omega_i\}$ occur in clusters of repeated, or nearly repeated, frequencies; in order to ensure asymptotic stability, as needed in the next section, we shall assume that all ω_i and ζ_i are positive. Defining the state vector $x = (\dot{\eta}_1,\omega_1\eta_1,\ldots,\dot{\eta}_n,\omega_n\eta_n)^T$ for this structure yields the state space representation $\dot{x} = Ax + Bu$, $y = Cx$, where $A = \mathrm{diag}(A_i)$, $B = (B_1^T,\ldots,B_n^T)^T$, and $C = (C_1,\ldots,C_n)$, with

$$A_i = \begin{pmatrix} -2\zeta_i\omega_i & -\omega_i \\ \omega_i & 0 \end{pmatrix}, \quad B_i = \begin{pmatrix} b_i \\ 0 \end{pmatrix}, \quad \text{and} \quad C_i = (c_{ri}, c_{di}/\omega_i) \tag{2}$$

Here b_i is the ith row of $\hat{B}$, and c_{ri} and c_{di} are the ith columns of $\hat{C}_r$ and $\hat{C}_d$, respectively. [Note that a somewhat similar modal representation was used in Ref. 6 for single-input/single-output structures. However, the simple form for A given by Eq. (10a) of Ref. 6 is based on a small-damping approximation; in general, the required representation would be considerably more complicated than Eq. (2).]

The problem we shall study is that of obtaining a reduced-order model

$$\dot{x}_r = A_r x_r + B_r u, \qquad y_r = C_r x_r$$

for this structure for which the output error

$$J = \|y - y_r\|_2 \tag{3}$$

is as small as possible for the specified model order n_r. This is termed the optimal reduction problem.[9] (For ease of interpretation it will often be preferable to deal with the normalized output error $\delta = J/\|y\|_2$.) Clearly, the optimal reduced-order model would have output error orthogonal to its output. To measure how close an actual reduced model comes to this ideal, we make use of the optimality index

$$\epsilon = <y - y_r, y_r> /(\|y - y_r\|_2 \|y_r\|_2) \tag{4}$$

defined in Ref. 11. Note that the optimal model of order n_r may give rise to an unacceptably large output error if n_r is set too low; conversely, if n_r is quite high, it is possible for a non-optimal reduced-order model to yield an error that is entirely acceptable. In practice, J and δ are probably of more direct interest than the optimality index ϵ.

The two techniques for model reduction that will be considered here are modal truncation and balancing.[4] If W_c and W_o are the controllability and observability grammians (see the next section), respectively, of the given system, then a balanced representation for it is simply[14] $\{T^{-1}AT, T^{-1}B, CT\}$, where T is the right eigenvector matrix of W_cW_o. This result can be used to derive a dominant reduced-order model for the system if some means is defined for deciding which balanced modes should be retained. Two measures that are commonly used are 1) Hankel singular values $\{\gamma_i\}$ (see Ref. 4), where the $\{\gamma_i^2\}$ are the eigenvalues of W_cW_o; and 2) component costs $\{\sigma_i\}$ (Refs. 2 and 3), where the $\{\sigma_i^2\}$ are the diagonal elements of $Z = -2AW_cW_o$. It is well established[1,5,6] that balanced and modal coordinates are approximately the same thing if damping is light and all frequencies are widely separated; this relation will now be rederived in a particularly straightforward way, using the results of Ref. 14. Furthermore, it will also be shown that this is *not* the case for the typical FSS with its closely spaced natural frequencies; balanced model reduction is then shown to be generally superior to model reduction based on modal truncation.

Closed-Form Grammians

The controllability and observability grammians, denoted by W_c and W_o, respectively, of the system described by Eq. (2) are the solutions of the algebraic Lyapunov equations,

$$AW_c + W_cA^T + BB^T = 0 \tag{5a}$$

and

$$A^TW_o + W_oA + C^TC = 0 \tag{5b}$$

The block diagonal form of A can be exploited[14] to give closed-form solutions for these positive-definite symmetric matrices. Taking W_c first and writing it in terms of its (2×2) blocks $\{W_{cij}\}$, we have

$$A_iW_{cij} + W_{cij}A_j^T + B_iB_j^T = 0 \tag{6}$$

Applying Eq. (2) then yields, after some algebra,

$$W_{cij} = \begin{bmatrix} 2\omega_i\omega_j(\zeta_j\omega_i + \zeta_i\omega_j) & \omega_j(\omega_j^2 - \omega_i^2) \\ -\omega_i(\omega_j^2 - \omega_i^2) & 2\omega_i\omega_j(\zeta_i\omega_i + \zeta_j\omega_j) \end{bmatrix} \times \beta_{ij}/d_{ij} \tag{7}$$

where $\beta_{ij} = b_ib_j^T$ and $d_{ij} = 4\omega_i\omega_j(\zeta_i\omega_i + \zeta_j\omega_j)(\zeta_j\omega_i + \zeta_i\omega_j) + (\omega_j^2 - \omega_i^2)^2$. The quantity d_{ij}^{-1} is essentially a measure of how closely correlated modes i and j are; it can be shown to be the square of the amplification factor between the input $\exp(\zeta_j\omega_jt)\cos(\omega_j\sqrt{[1-\zeta_j^2]}t)$, which excites mode j to infinity, and mode i.

It should be noted that an expression very similar to that of Ref. 14 was also independently derived by Skelton et al.[17] for use in modal cost analysis. This was obtained for the state space representation corresponding to $x = (\eta_1, \dot{\eta}_1, ..., \eta_n, \dot{\eta}_n)^T$; earlier expressions[2] were based on a state vector not so directly related to the modal parameters and as a result were much more complicated.

The general expression (7) for W_{cij} simplifies considerably for exactly repeated frequencies. In that case we obtain

$$W_{cij} = I_2 \times \beta_{ij}/2(\zeta_i + \zeta_j)\omega_i \tag{8}$$

In particular, the diagonal blocks are just $W_{cii} = I_2 \times \beta_{ii}/4\zeta_i\omega_i$. Similar simplifications occur for widely separated frequencies and $\zeta_i, \zeta_j \to 0$. In this case,

$$W_{cij} \to \begin{bmatrix} 0 & \omega_j \\ -\omega_i & 0 \end{bmatrix} \times \beta_{ij}/(\omega_j^2 - \omega_i^2) \tag{9}$$

Thus, W_c for a structure with light damping and all frequencies widely separated has an ith diagonal block proportional to $1/\zeta_i$ and off-diagonal blocks independent of $\{\zeta_i\}$ and thus tends to diagonal form for $\{\zeta_i\} \to 0$. While this asymptotic result is well known,[5,6,18] it is important to realize that it does not apply for the typical FSS case of repeated [Eq. (8)] or nearly repeated [Eq. (7)] natural frequencies. Balancing a modal representation of such a structure is consequently a nontrivial task.

The observability grammian W_o for a system with rate measurements only ($\hat{C}_d = 0$) can be obtained in a similar fashion, or more simply by noting that $A^T = PAP$, where $P = \mathrm{diag}(1, -1, ..., 1, -1)$. Therefore, pre- and postmultiplying Eq. (5) by P gives

$$A(PW_oP) + (PW_oP)A^T + C^TC = 0 \tag{10}$$

where use is made of the fact that $CP = C$ here. Thus, W_o is essentially as given by Eq. (7), with the only alterations being that the signs of the off-diagonal entries are changed and β_{ij} is replaced by $\gamma_{rij} = c_{ri}^Tc_{rj}$.

If displacement measurements are also allowed, the situation is much less simple; in fact, the analytical expressions that then result for W_o are really too complicated to be useful. The only exception to this is the expression for the ith diagonal block of W_o for a lightly damped structure ($\zeta_i \ll 1$), where we have the approximation

$$W_{oii} \approx I_2 \times (\omega_i^2\gamma_{rii} + \gamma_{dii})/4\zeta_i\omega_i^3 \tag{11}$$

with $\gamma_{dii} = c_{di}^Tc_{di}$. Although no general analytical expressions for W_o are viable, it is possible to derive a semi-closed-form method to evaluate this matrix that exploits the special form of $\{A,B,C\}$ in Eq. (2). For, let

$$W_{oij} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \tag{12}$$

be the (i,j)th block of W_o; then its defining equation [from Eq. (5)] $A_i^TW_{oij} + W_{oij}A_j + C_i^TC_j = 0$ can be expanded and rewritten as the system of linear equations

$$\begin{bmatrix} -2(\zeta_i\omega_i + \zeta_j\omega_j) & \omega_j & \omega_i & 0 \\ -\omega_j & -2\zeta_i\omega_i & 0 & \omega_i \\ -\omega_i & 0 & -2\zeta_j\omega_j & \omega_j \\ 0 & -\omega_i & -\omega_j & 0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} = -\begin{bmatrix} c_{ri}^Tc_{rj} \\ c_{ri}^Tc_{dj}/\omega_j \\ c_{di}^Tc_{rj}/\omega_i \\ c_{di}^Tc_{dj}/\omega_i\omega_j \end{bmatrix} \tag{13}$$

It is interesting to note that the determinant of the matrix in Eq. (13) is just d_{ij}; hence, this quantity plays a similar role in both the controllability and observability grammians.

As was true for W_c, the observability grammian of a lightly damped structure with widely separated natural frequencies can be shown to be block diagonally dominant. Thus, the Hankel singular values $\{\gamma_i\}$ for such a structure are approximately given as

$$\gamma_i^2 \approx \| W_{cii} \|_2 \| W_{oii} \|_2 = \beta_{ii}(\omega_i^2 \gamma_{rii} + \gamma_{dii})/(4\zeta_i \omega_i^2)^2 \tag{14}$$

a result given by Gregory.[5] This expression can then be used to obtain the $\{\sigma_i^2\}$ for the structure directly; these are given as the diagonal elements of $-2AW_cW_o \approx -2A\,\mathrm{diag}(\gamma_i^2)$. Thus from Eqs. (2) and (14) we have the two values

$$\sigma_{i1}^2 = 4\zeta_i\omega_i\gamma_i^2 \approx \beta_{ii}(\omega_i^2\gamma_{rii} + \gamma_{dii})/(4\zeta_i\omega_i^3), \qquad \sigma_{i2}^2 \approx 0 \tag{15}$$

for the ith balanced mode. Note that σ_{i1}^2 is precisely the ith modal cost of Skelton and Hughes.[1] In practice, the idealization (15) will not hold exactly; rather than σ_{i2} being precisely zero, we will have one large and one small value for each mode, with their mean $\hat{\sigma}_i$ constituting a good measure of overall modal cost. Again, it is important to remember that the approximations (14) and (15) are based on an implicit assumption of widely separated natural frequencies that is not valid for the typical FSS. For structures with clustered modes it is necessary to evaluate the off-diagonal blocks of W_c and W_o as well as the diagonal blocks considered above. This will be the subject of the next section.

Finally, if $p \geq m$ (typical of FSS applications) and there exists a matrix U with orthonormal columns satisfying $C = UB^TP$, then the system governed by Eq. (2) is said to be *orthogonally symmetric*,[19,20] and its cross-grammian W_{co} is defined as the solution of the Lyapunov equation

$$AW_{co} + W_{co}A + BU^TC = 0 \tag{16}$$

(Note that any FSS with *compatible*, i.e., physically collocated and coaxial, actuators and rate sensors and $C = B^T$ is necessarily an orthogonally symmetric system.) The usefulness of W_{co} for balancing and model reduction applications lies in the fact that it satisfies the relation $W_{co}^2 = W_cW_o$; thus, the $\{\gamma_i\}$ are just the absolute values of the eigenvalues of W_{co}. In fact, as $C^TC = PBU^TUB^TP = BB^T$ and $BU^TC = BU^TUB^TP = BB^T$, Eqs. (10) and (16) can be seen to reduce to the expressions[19] $W_{co} = W_cP = PW_o$. Thus, all three grammians of an orthogonally symmetric system are given directly from Eq. (7) with suitable changes of sign, noting, of course, that $\gamma_{rij} = \beta_{ij}$ and $\gamma_{dij} = 0$ for such systems.

Conditions for Near-Optimal Reduction

Conditions for near-optimal reduction, i.e., reduction resulting in a small output error δ, are considered in this section in both modal and balanced coordinates. Although both representations give similar results for small damping and widely separated natural frequencies, the results are sharply different for closely spaced frequencies.

Consider first modal truncation applied to an orthogonally symmetric[19,20] structural model. From Ref. 11 it follows that $\delta \ll 1$ if each (i,j) block W_{ij} of the cross grammian of this modal representation corresponding to a retained mode (i) and deleted mode (j) satisfies

$$\rho_{ij}^2 = \| W_{ij} \|^2/(\| W_{ii} \| \, \| W_{jj} \|) \ll 1 \tag{17}$$

Now, from Eq. (7) $W_{ij} = \beta_{ij}\tilde{W}_{ij}$ for $\tilde{W}_{ij}$ purely a function of ω_i, ω_j, ζ_i, and ζ_j; hence, the correlation coefficient ρ_{ij} can be written as a product

$$\rho_{ij} = \kappa_{ij}\tilde{\rho}_{ij} \tag{18}$$

where, from the Schwartz inequality, $\kappa_{ij}^2 = \beta_{ij}^2/(\beta_{ii}\beta_{jj}) \leq 1$, and $\tilde{\rho}_{ij}^2 = \| \tilde{W}_{ij} \|^2/(\| \tilde{W}_{ii} \| \, \| \tilde{W}_{jj} \|)$. The condition $\rho_{ij} \ll 1$ is now clearly

$$\kappa_{ij}\tilde{\rho}_{ij} \ll 1, \qquad i \neq j \tag{19}$$

Note that $\tilde{\rho}_{ij}$ depends on the structural properties ζ_i, ζ_j and ω_i, ω_j rather than on sensor/actuator locations; it reflects the relationship between structural parameters and the optimality of reduction. Since $0 \leq \kappa_{ij} \leq 1$, the condition $\tilde{\rho}_{ij} \ll 1$ certainly implies $\rho_{ij} \ll 1$, but not vice versa. In contrast, the coefficient κ_{ij} depends only on sensor/actuator locations. It can be seen that near-optimal reduction will occur if the ith and jth rows of B (columns of C) are nearly orthogonal for all retained modes i and deleted modes j. Thus, Eq. (19) decouples the contributions toward the optimality of reduction of the structural properties ($\tilde{\rho}_{ij}$) and sensor/actuator locations (κ_{ij}) in a very simple way.

We now study how the structural parameters ζ_i and ω_i influence the optimality index and in particular show that $\tilde{\rho}_{ij}$ can be quite large for the typical FSS case of nearly repeated natural frequencies. Introducing dimensionless variables $\alpha = \omega_i/\omega_j$, $\nu = \zeta_i/\zeta_j$, denoting for simplicity $\omega = \omega_j$, $\zeta = \zeta_j$, and introducing $\tilde{W}_{ij}$, $\tilde{W}_{ii}$, $\tilde{W}_{jj}$ from Eqs. (18) and (19), one obtains

$$\tilde{\rho}_{ij}^2 = n_{ij}/m_{ij}^2 \tag{20a}$$

where

$$m_{ij} = 4\zeta^2\alpha(\alpha + \nu)(1 + \alpha\nu) + (1 - \alpha^2)^2 \tag{20b}$$

and

$$n_{ij} = 8\zeta^2\alpha\nu\{4\zeta^2\alpha^2[(\alpha + \nu)^2 + (1 + \alpha\nu)^2] + (1 - \alpha^2)^2(1 + \alpha^2)\} \tag{20c}$$

Plots of $\tilde{\rho}_{ij}$ are shown in Fig. 1 for the special case $\zeta_i = \zeta_j = \zeta$, i.e., $\nu = 1$. The curves show that the condition of Eq. (19) is satisfied for small ζ and widely separated natural frequencies ($\alpha \neq 1$). In fact, the smaller the damping and the more separated the natural frequencies of the retained and deleted modes, the closer the reduction is to optimal. This confirms the well-known result of Refs. 5 and 6 that modal coordinates are approximately balanced, and thus (Ref. 4) uncorrelated, for such a structure. The importance of these graphs is that they apply for any structure: Once its $\{\zeta_i\}$ and $\{\omega_i\}$ are known, the $\{\tilde{\rho}_{ij}\}$ can simply be read off, giving important modal coupling information directly.

Consider now a general, rather than orthogonally symmetric, structural model. To analyze this, we must first form the product of the controllability and observability grammians,

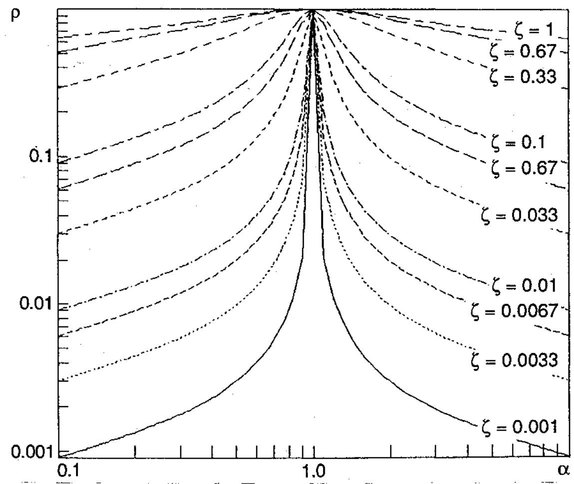


Fig. 1 Modal correlations for orthogonally symmetric structure.

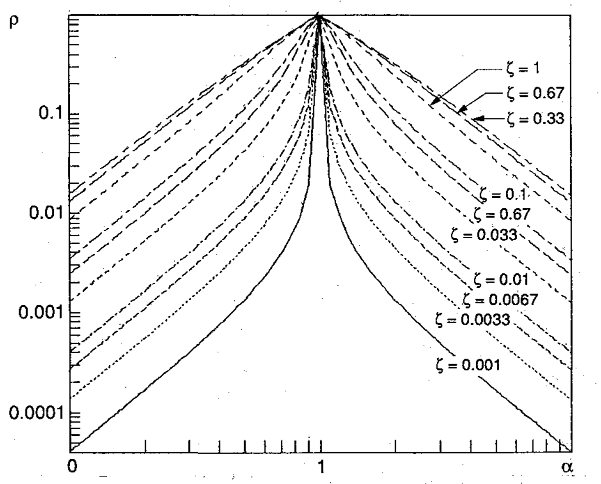


Fig. 2 Modal correlations for general structure, displacement outputs.

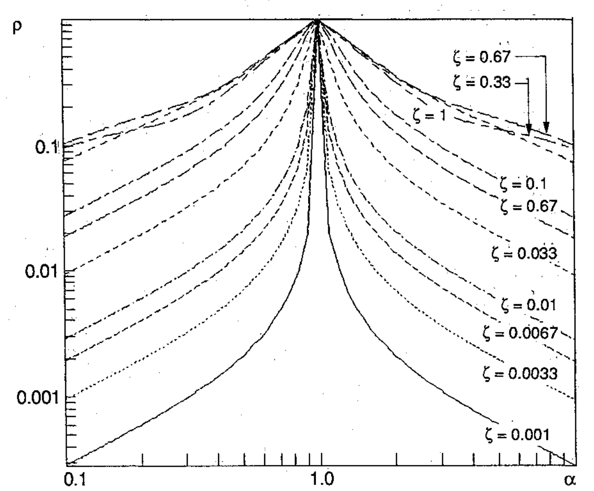


Fig. 3 Modal correlations for mixed rate and displacement outputs.

$W = W_c W_o$. Now, block-diagonal dominance of W is a necessary and sufficient condition for near-optimal reduction using modal truncation.[11] However, since it is difficult to determine the product of grammians in closed form, we use for simplicity a sufficient condition for near-optimality, namely, the block-diagonal dominance of both W_c and W_o. Defining the controllability and observability correlation coefficients [cf. Eq. (17)],

$$\rho_{cij}^2 = \|W_{cij}\|^2/\|W_{cii}\| \, \|W_{cjj}\| \tag{21a}$$

and

$$\rho_{oij}^2 = \|W_{oij}\|^2/\|W_{oii}\| \, \|W_{ojj}\| \tag{21b}$$

we can write for near-optimal reduction that $\rho_{cij} \ll 1$ and $\rho_{oij} \ll 1$, $i \neq j$. Thus, we have [cf. Eq. (19)]

$$\rho_{cij} = \kappa_{cij}\tilde{\rho}_{cij} \ll 1, \qquad \rho_{oij} = \kappa_{oij}\tilde{\rho}_{oij} \ll 1 \tag{22}$$

where

$$\kappa_{cij} = \kappa_{ij} \qquad \text{and} \qquad \kappa_{oij} = \|c_i^T c_j\|/(\|c_i\| \, \|c_j\|)$$

$$\tilde{\rho}_{cij}^2 = \|\tilde{W}_{cij}\|^2/(\|\tilde{W}_{cii}\| \, \|\tilde{W}_{cjj}\|)$$

and

$$\tilde{\rho}_{oij}^2 = \|\tilde{W}_{oij}\|^2/(\|\tilde{W}_{oii}\| \, \|\tilde{W}_{ojj}\|)$$

with

$$\tilde{W}_{cij} = W_{cij}/\beta_{ij} \qquad \text{and} \qquad \tilde{W}_{oij} = W_{oij}/\|c_i^T c_j\|$$

As was shown in the last section, the controllability grammian is essentially the same as the cross grammian in this case, up to the signs of its off-diagonal terms. Similar comments apply to the observability grammian if only rate measurements are used. Thus, the near-optimality conditions for the rate measurement case are the same as those obtained for orthogonally symmetric structures using the cross grammian. If displacement measurements are taken, though, the added complexity of the observability grammian [Eq. (13)] results in somewhat different values for the observability correlation coefficients. However, even now the same basic result applies that closely spaced poles result in correlated modes, especially for heavy damping. This is illustrated by the plot of $\tilde{\rho}_{oij}$ in Fig. 2 for the case of displacement measurements alone and the combined rate and displacement output example (for $c_{di}^T c_{dj} = c_{ri}^T c_{rj} = c_{di}^T c_{rj} = 1$ and $\omega_i = 1$) plotted in Fig. 3. [Of course, Eq. (22) implies that a near-optimal reduction can still be obtained, even for non-negligible $\tilde{\rho}_{cij}$ and $\tilde{\rho}_{oij}$, as long as κ_{cij} and κ_{oij} are small enough.]

These correlation results, confirming as they do the well-known results of Jonckheere and Gregory, show that model reduction using either balancing or modal truncation should yield approximately the same results for structures with light damping and sufficiently separated poles. This is confirmed by the following two examples.

Example 1

The vibrating system from Fig. 4 is investigated. The system has 3 DOF, two inputs and two outputs, the parameters $m_1 = m_2 = m_3 = 10$, $k_1 = k_4 = 1000$, $k_2 = 500$, $k_3 = 100$, $d_i = dk_i$, $i = 1,\ldots,4$, and

$$B^T = \begin{bmatrix} 0 & 0 & 0 & 0.1 & 0.2 & -0.5 \\ 0 & 0 & 0 & -0.2 & -0.3 & 0.1 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & -2 & 3 & 1 & -5 & -2 \\ 1 & 3 & -1 & 2 & -2 & 1 \end{bmatrix}$$

A near-optimal reduced system with 2 DOF is desired, using modal truncation.

The system poles for $d = 0.005$, $\lambda_{1,2} = -0.01823 \pm j6.0492$, $\gamma_{3,4} = -0.05544 \pm j10.529$, and $\lambda_{5,6} = -0.08627 \pm j13.135$ are widely separated. The system grammians are diagonally dominant, as expected, and the matrix $\Sigma = \text{diag}(\sigma_1,\ldots,\sigma_6)$ is $\Sigma = \text{diag}(0.5026,\ 3.8682,\ 4.8734,\ 0.1287,\ 0.0007,\ 0.1453)$. Deleting the last two rows of A and B and the last two columns of A and C, the cost of which is the smallest, we obtain the near-optimal reduced model. The optimality index of the obtained model is indeed smallest, $\epsilon = 0.001129$, and the reduction error is $\delta = 0.05767$. The solid line in Fig. 5 shows the dependence of the optimality index ϵ and the reduction error δ on the damping parameter d. As expected, the smaller the damping, the closer the reduced model is to optimal.

Example 2

The same system is now reduced in balanced coordinates. The matrices Γ and Σ are $\Gamma = \text{diag}(2.19310,\ 2.19305,\ 1.47585,\ 1.47529,\ 0.89912,\ 0.89898)$ and $\Sigma = \text{diag}(5.28370,\ 3.36072,\ 0.16177,\ 2.22386,\ 0.38322,\ 0.02391)$. A near-optimal model in balanced coordinates is obtained by deleting the last two columns of the transformed A and C and the last two rows of A and B, corresponding to the smallest cost. The poles of the reduced balanced model are $\lambda_{r1,2} = -0.01829 \pm j6.0492$, $\lambda_{r3,4} = -0.05541 \pm j10.529$. The model that is obtained is near-optimal, since its optimality index is $\epsilon = 0.67577 \times 10^{-5}$, and the reduction error is $\delta = 0.057661$. The dotted lines in

Fig. 5 show the dependence of ϵ and δ on the damping parameter d. From the plot of ϵ it can be seen that the smaller the damping, the closer the solution is to optimal. (In contrast, for high damping the output error δ falls off, indicating that one of the balanced modes becomes nearly unobservable.) Note that the results obtained for low levels of damping are very close to those obtained in the last example using modal coordinates. In contrast, for high damping the modal correlations increase to the point where balancing should be expected to give significantly better results than modal (Fig. 5 confirms this).

Balanced reduction should also give a reduced-order model that is closer to optimal than that produced by modal truncation, even for light damping, if the structure considered has closely spaced poles. The following pair of examples confirms that this is, in fact, true.

Example 3

The system from Fig. 4 is again considered, but now with the following parameters: $m_1 = m_3 = 1$, $m_2 = 2$, $k_1 = k_4 = 50$, $k_2 = k_3 = 5$, $d_i = 0.01\, k_i$, $i = 1, \ldots, 4$, and

$$B^T = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0], \qquad C = [1 \quad 2 \quad 2 \quad 0 \quad 0 \quad 0]$$

This system is again to be reduced from 3 to 2 DOF. The system poles are as follows: $\lambda_{1,2} = -0.02252 \pm j2.1224$, $\lambda_{3,4} = -0.27500 \pm j7.4111$, and $\lambda_{5,6} = -0.27748 \pm j7.4443$. The second and the third pair of poles are clearly closely spaced. The model reduction results obtained using modal truncation are

$$A_r = \begin{bmatrix} -0.0225 & 2.1224 & 0 & 0 \\ -2.1224 & -0.0225 & 0 & 0 \\ 0 & 0 & -0.2774 & 7.4443 \\ 0 & 0 & -7.4443 & -0.2774 \end{bmatrix},$$

$$B_r = \begin{bmatrix} -0.0002 \\ 0.0495 \\ -0.1580 \\ 0.6808 \end{bmatrix}$$

and $C_r = [1.0719 \; 0.0041 \; 0.2598 \; 0.0603]$, with $\epsilon = 0.7050$ and $\delta = 0.30634$. In this case the result is not near-optimal, but is the best one can obtain in modal coordinates, since deleting any other mode would lead to an even larger reduction error. The source of the difficulty is, of course, the close second and third pairs of poles. The correlation coefficients for these two modes, obtained from Eq. (17), are $\rho_{c23} = \rho_{o23} = 0.9964$, showing that the two modes are very highly correlated. The poor performance of this reduced-order model is further illustrated by the plots of the impulse response h and its power spectral density function S given in Fig. 6. The responses of the full (solid lines) and reduced (dotted lines) systems can be seen to be quite different.

Example 4

The same structure is now reexamined in balanced coordinates. The poles of the reduced system are $\lambda_{r1,2} = -0.02253 \pm j2.1224$ and $\lambda_{r3,4} = -0.27529 \pm j7.4630$ (i.e., the two close poles have now been separated during the reduction). The optimality parameters obtained are $\epsilon = 0.5423 \times 10^{-5}$ and $\delta = 0.001717$; hence, the reduction can be seen to be near-optimal. Indeed, the plots of impulse response and power spectral density function for this reduced system overlap those of the full system in Fig. 6. This example, taken together with the preceding one, illustrate the expected result that the high correlation of close modes prevents near-optimal model re-

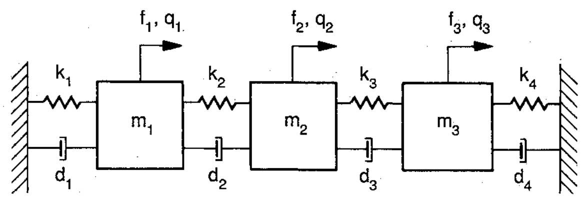
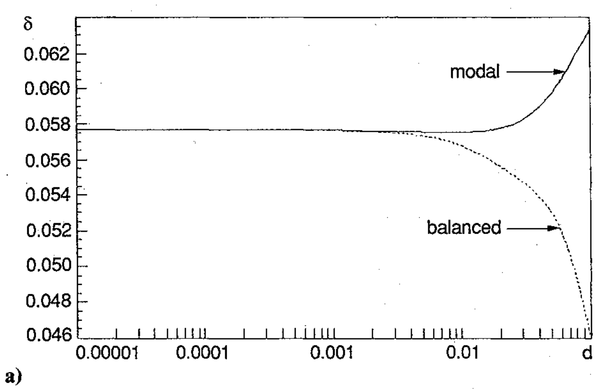


Fig. 4 Simple three-mode system.



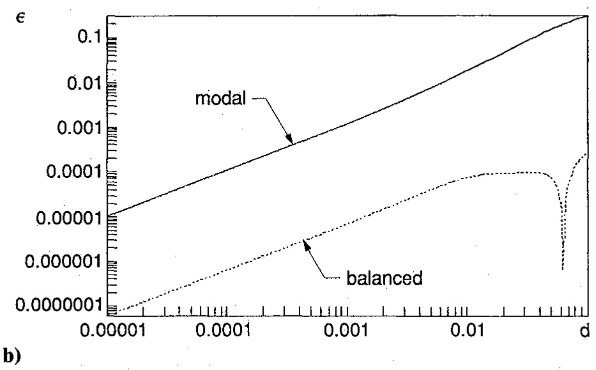
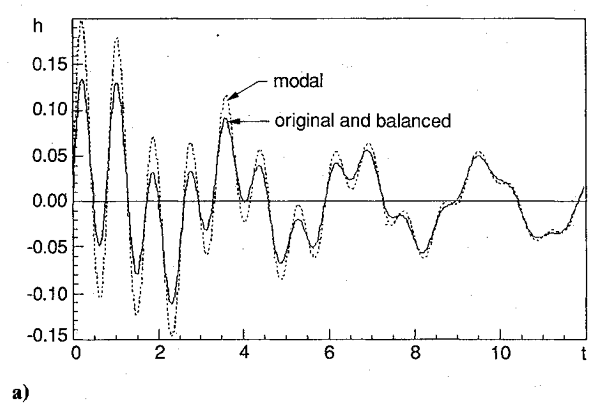


Fig. 5 Effect of damping on model reduction accuracy.



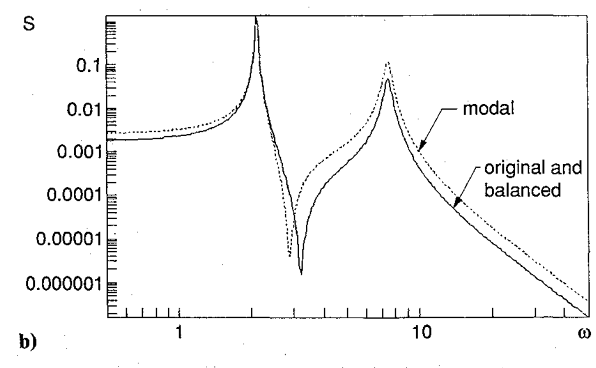


Fig. 6 Responses of full and balanced reduced-order models.

duction using modal truncation, but presents no such problems in balanced coordinates.

Robust Reduction for Flexible Structures

This section considers the question of developing a model reduction technique that is insensitive to damping variations in the structure considered. The component costs and Hankel singular values of a structure can be highly sensitive to changes in its damping; thus, the results of model reduction may be unreliable if the technique used is not robust to damping variations. In the next subsection the reduction index to be used is analyzed, and then robust reduction is considered.

Reduction Index

In modal coordinates the reduction index J is determined from (see Ref. 11)

$$J^2 = tr(C_t^T C_t W_{ct}) = tr(B_t B_t^T W_{ot}) = -2tr(A_t W_{ct} W_{ot}) \quad (23)$$

where the subscript t denotes the truncated (deleted) part of the system. For simplicity of notation this subscript is dropped in the following. If the grammians are diagonally dominant, we have

$$\| W_{cij} \|^2 \ll \| W_{cii} \| \, \| W_{cjj} \|, \qquad \| W_{oij} \|^2 \ll \| W_{oii} \| \, \| W_{ojj} \| \quad (24)$$

where W_{cij} and W_{oij} are the 2×2 matrices given earlier. (The particular norm used in these expressions is, of course, not very important; typically, the Frobenius norm is chosen for simplicity of computation.) If k is the number of retained variables, then from Eq. (23) we have

$$J^2 \approx \sum_{i=k+1}^{n} \sigma_i^2, \qquad \sigma_i^2 = tr(c_i^T c_i W_{cii})$$
$$= tr(b_i b_i^T W_{oii}) = -2tr(A_i W_{cii} W_{oii}) \quad (25)$$

where c_i is the ith column of C_t, b_i the ith row of B_t, and A_i the ith diagonal block of A_t, given by Eq. (2). In this case the value of the reduction index of the system is evaluated fairly accurately as a sum of the indices of each truncated mode.

Defining the correlation coefficients between the ith and the jth truncated modes,

$$\rho_{cij}^2 = \| W_{cij} \|^2 / \| W_{cii} \| \, \| W_{cjj} \|, \qquad \rho_{oij}^2 = \| W_{oij} \|^2 / \| W_{oii} \| \, \| W_{ojj} \| \quad (26)$$

the conditions of Eq. (24) are given as follows:

$$\rho_{cij} \ll 1 \quad \text{and} \quad \rho_{oij} \ll 1 \quad (27)$$

The conditions obtained in Eq. (27) are similar to Eq. (22), but here, instead of the correlation between a retained and a truncated mode, the correlation between two truncated modes is considered. Therefore, the conclusions from the preceding section apply here too: If the modal damping is small, and natural frequencies separated, or rows of the truncated part of B or columns of the truncated part of C are almost orthogonal, the reduction cost of two modes can be evaluated separately.

The value of each σ_i is determined as follows. From Eq. (15) we obtain

$$\sigma_i^2 = 0.5(\sigma_{i1}^2 + \sigma_{i2}^2) = 0.5\sigma_{i1}^2 = \beta_{ii}(\gamma_{dii} + \omega_i^2 \gamma_{rii})/(8\zeta_i \omega_i^3) \quad (28)$$

Let Y_i denote the amplitude of the output for $\omega = \omega_i$ which is approximately the resonance amplitude of the ith mode:

$$Y_i^2 = \beta_{ii}(\gamma_{dii} + \omega_i^2 \gamma_{rii})/(4\zeta_i^2 \omega_i^4) \quad (29)$$

Then, comparing Eqs. (28) and (29), the reduction cost of the ith mode is obtained as

$$\sigma_i = 0.5 Y_i \sqrt{\Delta_i} \quad (30)$$

where $\Delta_i = 2\zeta_i \omega_i$ is the half-power frequency (see Ref. 21). The last formula shows that the approximate reduction cost of the ith mode is proportional to both the square of the amplitude of the ith resonance and the width of this resonance.

Robust Reduction

The value of the reduction index σ_i (reduction cost) determines whether the ith mode should be retained or deleted. For flexible structures the cost is determined from Eq. (30), and its value is inversely proportional to the modal damping. However, for flexible structures damping is the least accurately known variable: Its value can vary in quite a wide range, even more than 100% of its nominal value. For this reason, and since the reduction cost is very sensitive to damping variations for small damping, reduction can be quite unreliable. As a result, if a part of the system is deleted on the basis of the reduction costs obtained for the nominal damping level, there is no guarantee that the resulting reduction error will be small for the true structural damping. In this subsection we present an approach that can help to solve this problem.

Denote the nominal modal damping by ζ_{in}, and in its lower and upper variations by $\Delta\zeta_i^- (\geq 0)$ and $\Delta\zeta_i^+ (\geq 0)$, respectively, so that $Z_i = [\zeta_{in} - \Delta\zeta_i^-, \zeta_{in} + \Delta\zeta_i^+]$ is a damping variation interval, and

$$\mu^+ = \max_i (\Delta\zeta_i^+ \omega_i), \qquad \mu^- = \max_i (\Delta\zeta_i^- \omega_i) \quad (31)$$

The upper and lower bounds of cost are accurately determined by shifting all system poles by $\mu = \mu^-$ and $\mu = -\mu^+$. For the shifted poles, the grammians are obtained from

$$(A + \mu I)W_{\mu c} + W_{\mu c}(A^* + \mu I) + BB^* = 0$$
$$(A^* + \mu I)W_{\mu o} + W_{\mu o}(A + \mu I) + C^*C = 0 \quad (32)$$

In terms of flexible structures the shift to the right, for $\mu = \mu^-$, means subtracting structural damping, whereas the shift to the left, for $\mu = -\mu^+$, means adding structural damping.

The following robust reduction procedure can then be applied:

1) Determine μ^- and μ^+ from Eq. (31) for given lower ($\Delta\zeta_i^-$) and upper ($\Delta\zeta_i^+$) damping estimates, and from Eq. (32) find the grammians W_c^+, W_o^+, and W_c^-, W_o^-, respectively.

2) For upper (W_c^+, W_o^+) and lower (W_c^-, W_o^-,) grammians

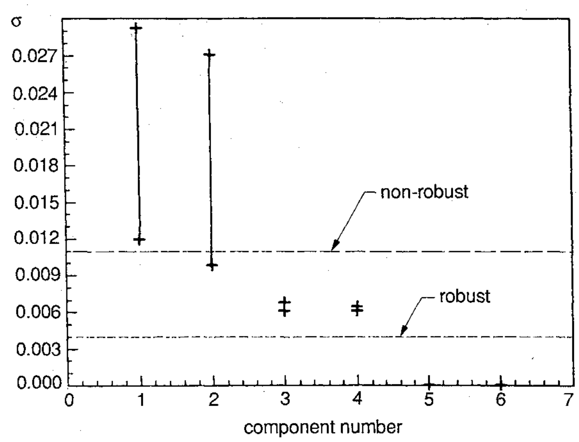


Fig. 7 Modal cost uncertainties for a three-mode system.

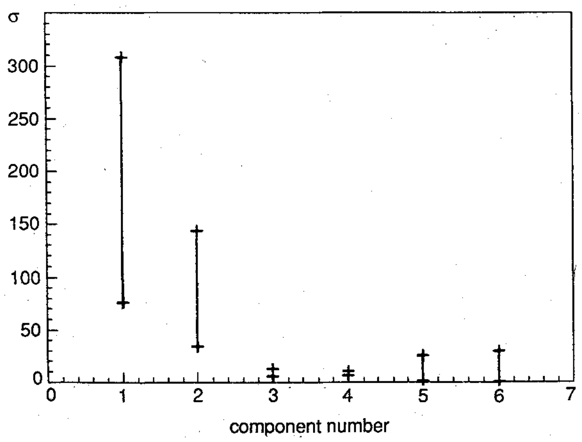


Fig. 8 Modal cost uncertainties for a three-mode system.

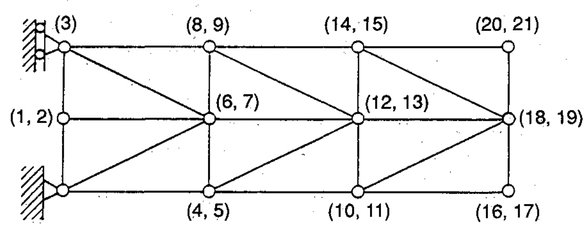


Fig. 9 Flexible truss structure.

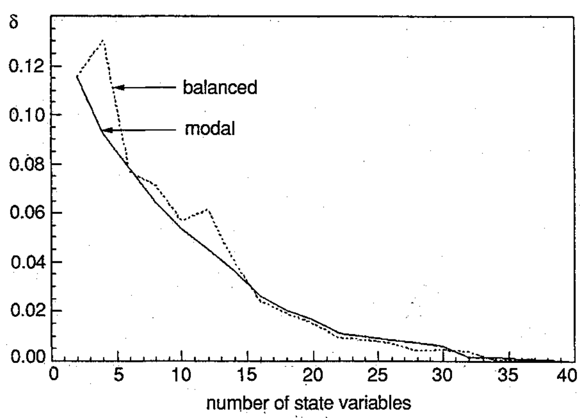


Fig. 10 Effect of model order on truss reduction error.

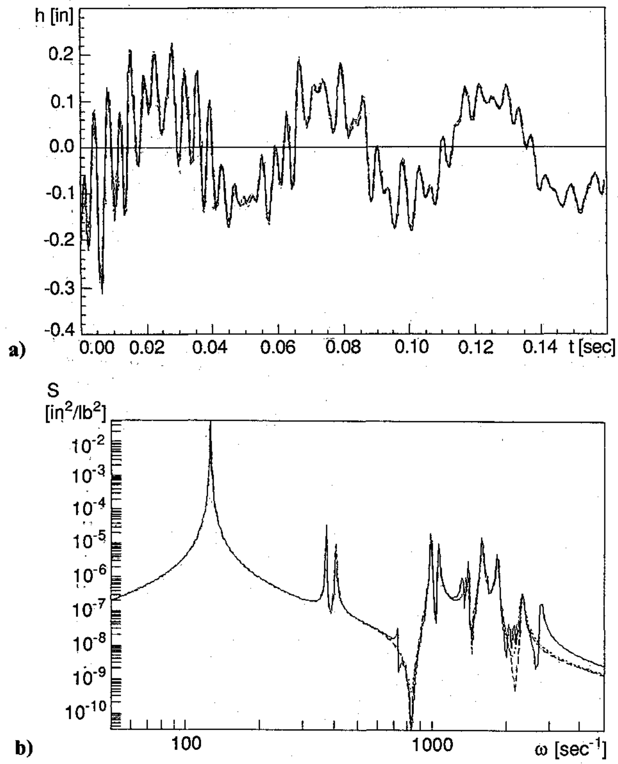


Fig. 11 Responses of full and reduced truss models.

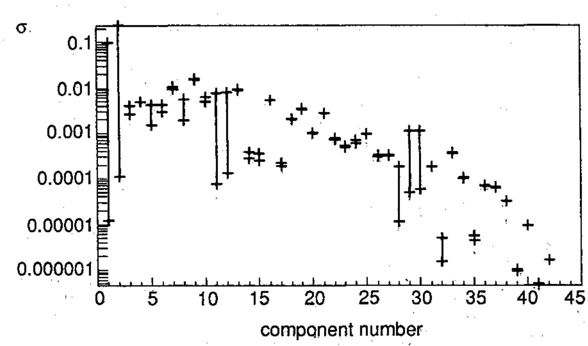


Fig. 12 Modal cost uncertainties for the flexible truss.

determine the upper and lower costs (σ_i^+ and σ_i^-) and the cost interval $\Delta\sigma_i = [\sigma_i^-, \sigma_i^+]$.

3) Plot the cost interval of each component (see Fig. 7 for a typical case). If the cost intervals of the truncated parts of the system do not overlap the cost intervals of the retained parts of the system (dotted line in Fig. 7), then the reduction is robust with respect to the damping variations: Any value of damping taken from the intervals Z_i gives the same reduced model. If the cost intervals of the truncated and retained subsystems overlap (dashed line in Fig. 7), the reduction is nonrobust: Its results depend on the particular damping values from the intervals Z_i.

Example 5

A balanced reduction of the system from example 3 is considered. Its damping is not exactly known, and we assume that the system modal damping variations are $\mu^+ = \mu^- = 0.01$. The cost bounds obtained for this case are displayed in Fig. 7, showing that the reduction from 6 to 4 (or 2) states is robust.

Example 6

A balanced reduction for the system from Fig. 4, for $m_1 = 4$, $m_2 = 2$, $m_3 = 7$, $k_1 = 10$, $k_2 = k_3 = 15$, $k_4 = 20$, damping $d_i = 0.005k_i$, $i = 1,...,4$, and

$$B^T = (0 \;\; 0 \;\; 0 \;\; 1 \;\; 2 \;\; -1), \qquad C = \begin{bmatrix} 10 & 2 & 2 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

is considered. The system poles are $\lambda_{1,2} = -0.0056 \pm j1.4913$, $\lambda_{3,4} = -0.0138 \pm j2.3529$, and $\lambda_{5,6} = -0.0462 \pm j4.2997$. The lower bound for modal damping perturbation is $\mu^- = 0.005$, and the upper bound is $\mu^+ = 0.003$. The bounds of the reduction cost are shown in Fig. 8. The reduction from 6 to 4 states, based on the nominal damping levels, retains states 1, 2, 5, and 6, while deleting states 3 and 4. One can see from Fig. 8 that this reduction is not robust.

Applications

In this section reduction of a flexible truss model and the COFS − 1 mast with sensor/actuator dynamics are presented. The truss is shown in Fig. 9. Every truss node has 2 DOF, as numbered in Fig. 9. The first number denotes the horizontal displacement and the second the vertical displacement. Let f_i be the force applied to the ith degree of freedom, and q_i and v_i be the displacement and velocity of this point, respectively. The system has one independent input u applied at four

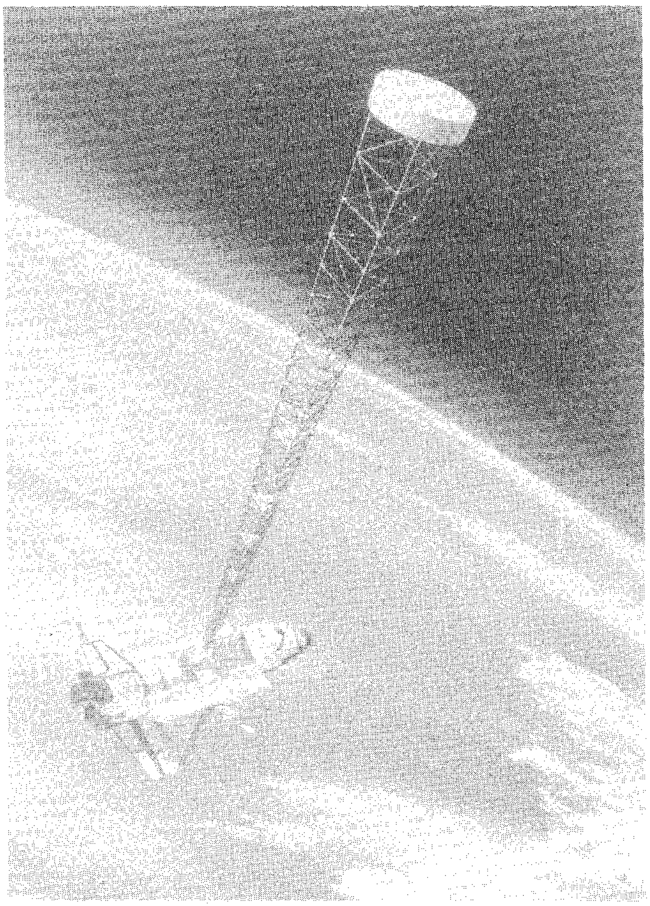


Fig. 13 Proposed COFS-1 flight experiment.

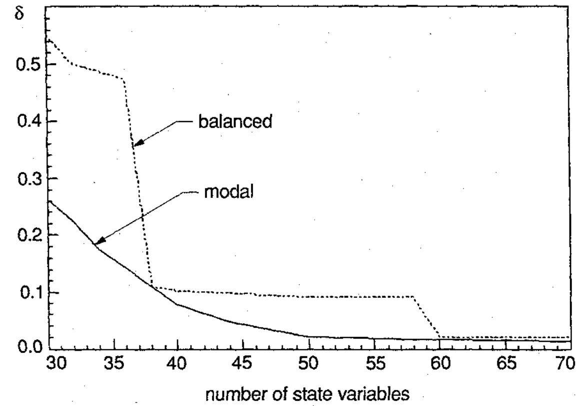


Fig. 14 Effect of model order on COFS-1 reduction error.

locations, as follows: $f_{11} = f_{12} = f_{21} = -f_{14} = 1$. Similarly, one measured output is constructed from the displacements at four points: $y = q_{15} + q_{16} + q_{17} + q_{20}$.

The truss model was reduced in balanced and modal coordinates. The results are presented in Fig. 10 (balanced reduction, dotted line; modal reduction, solid line). It shows that the reduction from 42 to 4 states causes reduction error δ less than 10%, both in balanced and modal coordinates. The plots of the impulse response h and power spectrum S are shown in Fig. 11. Next, the robustness of the truss reduction was tested. The lower variation of damping is $\mu^- = 0.01$, and the upper variation is $\mu^+ = 3$. The cost bounds are plotted in Fig. 12, which shows that the reduction in most cases is not robust.

The model of the COFS-1 structure (Fig. 13) consists of its structural properties as well as its actuator and sensor dynamics. It has 104 state variables, 10 inputs, and 20 outputs. The reduction results, in balanced as well as modal coordinates,

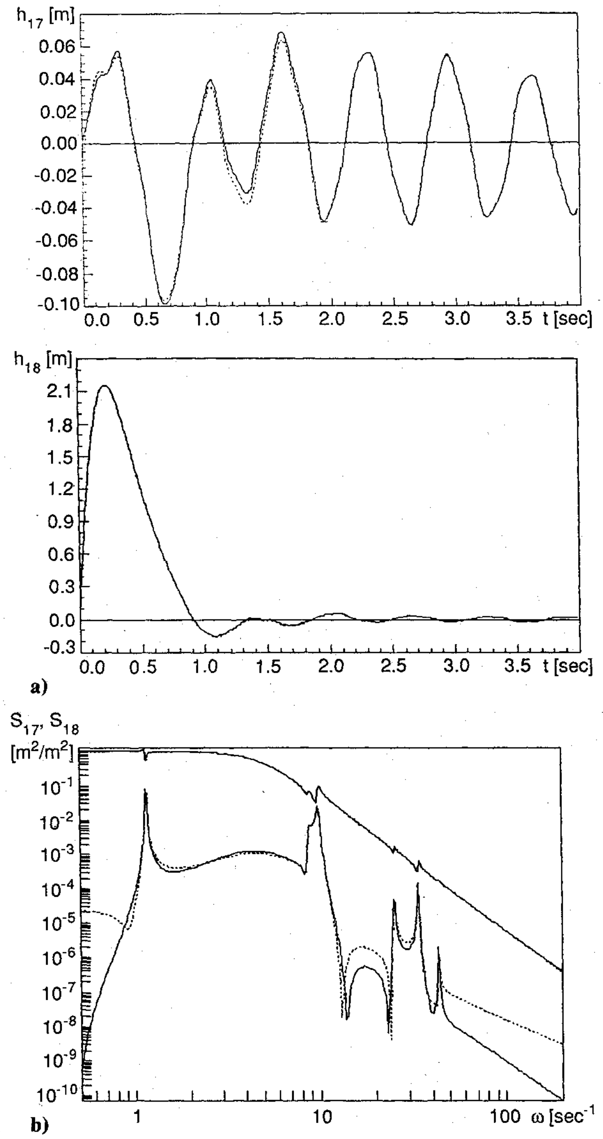


Fig. 15 Responses of full and reduced COFS-1 models.

are presented in Fig. 14. They show that reduction from 104 to 38 states causes an error about 10%, and to 60 states an error about 2%, in both balanced and modal coordinates (balanced reduction, dotted line; modal reduction, solid line). Plots in Figs. 15a and 15b show the impulse responses and power spectra of the full and reduced (60 state) model, with input 8 and outputs 17 and 18.

Conclusions

This paper has considered the problem of model reduction for flexible structures by means of either modal truncation or the technique of balancing. Although it is well established that these methods are essentially the same for a lightly damped structure with widely separated natural frequencies, it was emphasized that the same is *not* true for the typical flexible space structure with its many closely spaced resonances. These considerations were quantified by using the concept of modal correlations, which were derived from closed-form expressions for the grammians of a structure and displayed in a particularly simple graphical form. Another question addressed was that of the sensitivity of modeling error to variations in the (generally ill-defined) damping present in the structure; a robust model reduction technique was developed to cope with this. Finally, a flexible truss and the COFS-1 mast were used to give realistic applications for the model reduction techniques studied.

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